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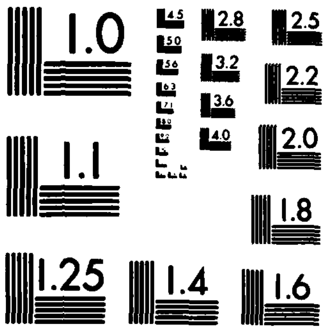

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SCHOOL OF INDUSTRIAL AND SYSTEMS ENGINEERING

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A SIMPLE AND EFFICIENT ALGORITHM TO COMPUTE TAIL PROBABILITIES FROM TRANSFORMS

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Abstract

We present an algorithm to compute the "tail" probability that a random variable exceeds a specified number, given only an expression for its transform. Our method consists essentially of summing a power series, so it is easy to perform and requires little memory. Furthermore, any desired degree of accuracy may be specified in advance of the computation, after which the computational effort is nearly linear in the reciprocal of the prespecified error. We also show that the problem is NP-hard, suggesting that there exists no procedure significantly better than ours.

KEY WORDS include: Tail probability, Laplace transform, computational complexity, fully polynomial approximation scheme, convolution.

OR/MS Subject Classification: Major 564 (probability/distributions); minor 432 (mathematics/combinatorics).

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A *tail probability* expresses the likelihood that a random variable will exceed some specified threshold. Tail probabilities are important to system designers because values exceeding certain thresholds may represent unacceptable or even catastrophic system behaviors. This paper investigates the problem of obtaining the tail probability of a random variable whose *transform* is given explicitly, but whose distribution is not necessarily known. This problem typically arises when the random variable of interest is a function of given "elementary" system variables. Applications include:

(1) Reliability analysis, where the probability of system failure, a function of known individual component reliabilities, must not exceed a given tolerance, and

(2) Queuing systems, where the probability of buffer overflow or excessive waiting time is sought.

Unfortunately, tail probabilities can be difficult to compute, especially when the random variable's distribution is not available in closed form. The standard "transform inversion formula" is an analytical expression (an integral over an infinite range), not an algorithm. Chebyshev bounds (on the tail probability) are easily obtained from a transform, but are rarely tight enough to be useful.

Various methods have been proposed for the related problem of recovering the *density* of a random variable from its transform. But computing a tail probability by first computing the density entails new difficulties: the density may not exist; many density evaluations are needed to perform the tail integral. Furthermore, these methods are difficult to implement and few include numerically computable bounds on error and computational effort. Many of the techniques surveyed by Piessons (1975) and Piessons and Dang (1976) involve solving large, ill-conditioned systems of linear equations. Jagerman (1978) offered a method based on the derivatives of the transform. While this paper was in preparation, Honig and Hirdes (1984) published a method similar to, but weaker than ours.

In this paper, we introduce a more direct algorithm to compute tail probabilities from transforms. Our method is both faster and simpler than others previously proposed. It requires virtually no memory and consists essentially of summing a power series. It also permits the computational effort and worst-case error to be specified *in advance* - the smaller the acceptable error, the greater the effort required. Using the methods of computational complexity theory [1], we show that our method is a *fully polynomial approximation scheme* and we also offer a strong argument that there is no significantly better algorithm by showing a formalization of the problem to be NP-hard. Finally, we provide numerical examples to illustrate the manner in which errors are prespecified and the ease with which approximate solutions are obtained.

Our method can also be used to obtain, from its transform, the probability that a random variable will fall within any given interval of the real line. It can also be modified (in a straightforward way) to provide strict upper or lower bounds on probabilities, rather than arbitrarily close estimates. However, we have restricted our discussion to approximately computing (upper) tail probabilities because (1) in our experience, most applications call for good estimates of tail probabilities; and (2) tail probabilities are small (typically between 10^{-2} to 10^{-9}) and so are especially vulnerable to distortion by the roundoff errors of floating-point arithmetic. Accordingly, we will refer to the complementary cumulative distribution function $P[X>\cdot]$ as the "tail probability function."

1. THE PROBLEM AND OUR METHOD FOR SOLVING IT

Since most transforms are interchangeable with minimal computational effort, we will focus our attention on the familiar *(bilateral) Laplace transform*, expressed as a function of the argument s, and defined by

$$\mathcal{L}_X(s) = E[e^{-sX}], \tag{1.1}$$

where X is the random variable whose distribution is "transformed." It is customary to view s and $\mathcal{L}(s)$ as *complex* numbers.

The random variable X may be continuous or discrete; we simply require that the expectation (1.1) exists for any imaginary s (i.e., any s whose real part is zero). All distributions of practical interest satisfy this criterion, notably those, such as the waiting time in a queueing system, that are partly continuous and partly discrete, and to which conventional methods of transform inversion are inapplicable.

The appeal of transforms is based on the fact that many simple operations on random variables correspond to simple operations on their transforms, while their distributions undergo more complicated transformations. For example, if $X = \sum_i X_i$ is a sum of independent random variables, then

$$\mathcal{L}_X(s) = E\left[e^{-s \sum_i X_i}\right] = \prod_i E\left[e^{-sX_i}\right] = \prod_i \mathcal{L}_{X_i}(s), \tag{1.2}$$

that is, the transform of X is merely the product of the transforms of the components, X_i. By way of comparison, the *density* of X is the *convolution* of the components' densities, and is much harder to compute.

Suppose that we are given a procedure to compute $\mathcal{L}_X(s)$ for any imaginary argument s. Our problem is to compute the tail probability P[X>A], where A is a given parameter. We propose to compute an *approximate* tail probability, τ, according to the finite summation

$$\tau = \frac{U-A}{U-L} + \sum_{n=1}^{N} \frac{\alpha^{n^2}}{n\pi} \cdot \mathrm{im}\left[(\beta^n - \gamma^n) \cdot \mathcal{L}_X(-n\omega j) \right], \tag{1.3}$$

where im[·] denotes the imaginary part of a complex number, and

$$\alpha = e^{-D^2\omega^2/2}, \qquad \beta = \exp(-A\omega j) = \cos(A\omega) - j\sin(A\omega),$$
$$\gamma = \exp(-U\omega j) = \cos(U\omega) - j\sin(U\omega), \qquad \omega = \frac{2\pi}{U-L}, \qquad j = \sqrt{-1}. \tag{1.4}$$

Note that τ is a function of the transform $\mathcal{L}(\cdot)$ and the threshold A, as well as four parameters: L, U, D and N. We will show that these parameters can be chosen to make τ as close as desired to the true solution P[X>A]. The computational effort is essentially determined by N. We discuss the effort and error associated with this approximation in the two sections that follow.

2. COMPUTATIONAL REQUIREMENTS

The standard measures of computational effort are number of elementary operations (addition, subtraction, multiplication or division) performed and number of memory "registers" utilized. To obtain these measures, we must explicitly consider the precision to which operations are performed, and how values of the transform are obtained. To this end, we introduce two technical assumptions:

(A1) All operations, as well as evaluations of the transform, are performed to k significant decimal digits.

(A2) The value of $\mathcal{L}_X(s)$ is obtained in a finite number of operations that do not depend on s.

The sum (1.3) is robust with respect to truncation/roundoff errors since none of its terms exceeds 1 in absolute value. Thus, direct evaluation of τ according to (1.3) and (A.1) results in an error of at most $\pm N\ 10^{-k}$.

There is, however, a faster way to evaluate τ: the powers of α, β and γ in (1.3) may be generated recursively. Seven memory registers will suffice; they contain (1) the cumulative sum to date, (2) α^{2n+1}, (3) α^{n^2}, (4-5) the complex pair β^n, and (6-7) the complex pair γ^n. Note that this memory requirement is small and does not depend on N.

Transform evaluations may be accelerated in a similar way if there holds

(A.2') $\mathcal{L}_X(s)$ may be expressed as a finite number of operations upon s and functions of s taking the form e^{cs^m} for some number c and some integer m.

Clearly (A.2') $\Rightarrow$ (A.2). But if (A.2') holds, the trigonometric functions in $\mathcal{L}_X$ may be computed once and then recursively generated, as described above. This substantially decreases the number of operations required to evaluate the transform, and it increases the memory requirement only by a finite quantity that depends on $\mathcal{L}_X$ but not N.

Since each term in (1.3) requires a finite number of elementary operations (including updating the powers of α, β and γ, and evaluating the transform), only O(N) operations are required to compute τ.

3. ERROR ANALYSIS

We now consider how close τ will be to the desired tail probability $P[X>A]$. Let Z be a zero mean, unit variance, normally distributed random variable independent of X, and define

$$\theta = P[X-DZ-L \bmod U-L > A-L \bmod U-L], \tag{3.1}$$

where L, U and D are given numbers such that $L << A << U$ and $D/(U-L) << 1$. Intuitively, L and U represent lower and upper limits (respectively) on the effective range of X (that is, $P[L < X < U] \sim 1$) and D represents the accuracy to which X and A are specified. We will view θ as an approximation to $P[X>A]$. To make θ close to $P[X>A]$, we must make D very small (i.e. positive but near zero), L of large negative magnitude and U of large positive magnitude.

In Appendix A, we show that θ is exactly determined by the infinite sum

$$\theta = \frac{U-A}{U-L} + \sum_{n=1}^{\infty} \frac{\alpha^{n^2}}{n\pi} \cdot \mathrm{im}\left[(\beta^n - \gamma^n) \cdot \mathscr{L}_X(-n\omega j) \right]. \tag{3.2}$$

Note that (1.3) contains only the first N terms of (3.2).

For clarity of presentation, we break the error by which τ and $P[X>A]$ may differ into three components, the *accuracy* E_a, the *precision* E_p, and the *truncation* E_t. These variables will serve as error bounds in the sense that

$$\begin{gathered} P[X \ge A + E_a \cdot (U-L)] - E_p \le \theta \le P[X>A-E_a \cdot (U-L)] + E_p , \\ \theta - E_t \le \tau \le \theta + E_t . \end{gathered} \tag{3.3}$$

Loosely speaking, (3.3) states that τ must lie within E_t of θ, and θ must lie within E_p of a probability in the range from $P[X \ge A']$ to $P[X>A']$, where A' differs from A by less than a fraction E_a of the effective range of X. This is illustrated in Figure 1. Accuracy is expressed as a fraction of the range U-L of X, so that E_a, like the other errors, is dimensionless. (A fourth source of error, the roundoff caused by computing to k decimal digits, was discussed in Section 2.)

The error bounds E_* and the parameters D and N in (1.3)-(1.4) will be expressed as functions of L, U, and three additional variables: r, z_p and z_t. The first, called *resolution*, specifies the desired accuracy:

$$r = \frac{D}{U-L}. \tag{3.4}$$

The second, z_p, expresses the desired precision in the sense that E_p should be comparable to $\bar{\Phi}(z_p)$, where

$$\bar{\Phi}(z) = P[Z>z] = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2} dx \tag{3.5}$$

is the familiar tail function associated with the normal distribution. The third, z_t, similarly expresses the desired truncation error.

No closed-form expression for the integral (3.5) exists, and the tables in most statistics texts do not extend into the range of interest to us here. So we have provided a graph of $\bar{\Phi}$ in Figure 2. It shows, for example, that z=6 corresponds to the precision 10^{-9} .

The exact relationship between r, z_p and z_t and the errors E_* is given by

Theorem 1. The bounds (3.3) are satisfied when

$$E_a = rz_p , \tag{3.6}$$

$$E_p = \max(P[X<L+Dz_p], P[X>U-Dz_p]) + 2\,\bar{\Phi}(z_p), \tag{3.7}$$

$$E_t = \frac{1.6}{z_t}\,\bar{\Phi}(z_t), \tag{3.8}$$

and

$$z_t = 2\pi rN. \tag{3.9}$$

Proof. See Appendix B.

If L and U are sufficiently far apart, (3.7) and (3.8) imply that the errors E_p and E_t may be made extremely small by selecting modest values of z_p and z_t (see Figure 2). Specifically, E_p and E_t can be made within 10^{-k} (that is, negligible when k-digit decimal arithmetic is used) by choosing z_p and z_t to be O(k). Moreover, for fixed z_p , (3.6) shows that the accuracy E_a is proportional to the resolution r. But (3.6) and (3.9) combine to form

$$E_a \cdot N = \frac{z_p \cdot z_t}{2\pi}. \tag{3.10}$$

Thus we cannot simultaneously guarantee fast execution and good accuracy; the choice of r will dictate the desired tradeoff between these two aspects of the heuristic's performance.

If we neglect errors on the order of k-digit roundoff, (3.10) shows our method to be a fully polynomial (indeed, linear) approximation scheme [1].

4. INTUITIVE MOTIVATION

Our method was inspired by the concept of *bandwidth* as a measure of the inherent complexity of a function, and the notion that by limiting bandwidth, essential form may be retained while complexity is reduced. Modern communications systems utilize this technique to transmit approximate signals that require only a fraction of a line's transmitting capacity; in this way, thousands of signals may be simultaneously transmitted on a single line.

The concepts of *bandwidth* and *transform* are closely related. Any signal may be viewed as a linear combination of sinusoidal components. A transform expresses the magnitude of each component, and the bandwidth is the range of component frequencies contained in the signal. A limited-bandwidth signal approximation is obtained by erasing sections of the transform of (intuitively, removing part of the complexity in) the original signal. If we eliminate the components whose frequencies are too small or too large to be of practical concern, we are left with a finite band of frequencies that express the signal's essential characteristics.

In computing tail probabilities from transforms, we view the probability density as a signal to be approximated. Every evaluation of the transform yields information about this density, and we wish to make do with as few evaluations as possible. So we eliminate low-frequency components by neglecting the density outside the range L to U, and we eliminate high-frequency components by permitting A to be displaced by a small multiple of D. More specifically, restricting the probability distribution to the range [L, U] restricts its frequency components to integer multiples of ω, and adding -DZ to X multiplies the transform of X by a transform whose values decrease rapidly as the frequency increases beyond 1/D. We are left with an approximate density whose tail probability is easily computed.

5. COMPLEXITY ANALYSIS

We now establish the difficulty of computing tail probabilities *exactly*, even from very simple transforms. We formalize the problem as follows: Evaluate $P[X>A]$ given a number A and a transform $\mathcal{L}_X(s)$ expressed in closed form, that is, as a finite word generated (according to the grammar of complex arithmetic) by real numbers and the symbols s, e, +, −, *, /, ↑, (, and).

We show that this problem is at least as difficult (in the worst case) as the following knapsack problem: Given $M+1$ positive integers $t_1, \ldots, t_M$, and A, is there a subset S of $\{1,\ldots,M\}$ such that $\sum_{i \in S} t_i = A$? This problem is known to be NP-complete [1]. Hence,

Theorem 2. Computing tail probabilities from transforms is NP-hard.

Proof. Let $X = \sum X_i$, with the X_i mutually independent and

$$X_i = \begin{cases} t_i, \text{ with probability } \frac{1}{2} . \\ 0, \text{ with probability } \frac{1}{2} . \end{cases}$$

If we could quickly compute $P[X>A-1]$ and $P[X>A]$, then we could also quickly compute

$$P[X>A-1] - P[X>A] = P[X=A] = 2^{-M} \cdot \# \left\{ S \subseteq \{1,\ldots M\} \;\middle|\; \sum_{i \in S} t_i = A \right\}$$

to see whether it is non-zero, and so determine whether the knapsack problem has a solution. But the knapsack problem is NP-hard, so the computation of tail probabilities must be NP-hard as well. ∎

In this sense, the computation of tail probabilities from transforms resembles the knapsack problem, for which fully polynomial approximation schemes are also available [Lawler (1979)]. Note, however, that our definition of the error E_a refers to a perturbation of the parameter A, whereas the more common definition of error refers to a perturbation of the result, τ. This distinction is necessary because we are dealing with real numbers, not integers, and because the tail function P[X>A] may be discontinuous.

Like many complexity results, Theorem 2 states that some instances of a problem are hard to solve. Of course, other instances are easy to solve, for example, problems involving transforms of uniformly distributed random variables. It is important therefore to consider whether there are large subclasses of problems for which the conclusion of Theorem 2 fails to hold.

The proof of Theorem 2 rests on a connection between certain discrete random variables and the knapsack problem. When X is discrete, the tail probability P[X>A] is discontinuous in A, and this obviously makes the task of computing tail probabilities more difficult. One might be tempted to conclude that discontinuity alone is responsible for the conclusion of Theorem 2, i.e., that only the tail probabilities of discrete random variables are difficult to obtain. But this is not so: The random variables whose tail probabilities are sought can be restricted to have bounded density, yet the result of Theorem 2 still holds; simply redefine the random variables X_i in the proof to be uniformly distributed over $[0, 1/M] \cup [t_i, t_i+1/M]$. Similarly, the conclusion of Theorem 2 can be shown to hold even if the density is required to be continuous, or continuously differentiable.

Thus, the class of transforms from which tail probabilities are hard to obtain is nontrivial, and Theorem 2 implies (under the customary complexity assumption P≠NP) that no procedure exists that will quickly compute tail probabilities from general transform specifications. This is strong circumstantial evidence that there can be no significantly better method than the one we have proposed.

6. NUMERICAL EXAMPLES

We now provide numerical examples to illustrate the advantages of our approach. Four random variables will be considered. In each case, we will seek to determine the tail probability that X exceeds $A = \mu + 4\sigma$, where μ and σ denote the mean and standard deviation (respectively) of X. These random variables are as described below. A summary of problem parameters is given in Table 1. Note that each of the transforms shown in Table 1 satisfies the assumption (A.2') defined in Section 2.

PROBLEM A. X_A is a normal random variable of zero mean and unit variance. We seek to compute $\bar{\Phi}(4)$. The exact solution is available in "probability tables." However, it should be noted that no closed-form expression for the integral (3.5) exists, and such tables are not easy to compute!

PROBLEM B. X_B is a binomial random variable representing the number of heads obtained in 100 flips of a fair coin. We seek the probability of more than 70 heads. The exact solution is obtained by summing the discrete probabilities of 71 through 100 heads. As predicted by the Central Limit Theorem, the solution to problem B is close that of problem A.

PROBLEM C. X_C denotes the total earnings from twenty games, where the probability of winning the m-th game is $1/\sqrt{m}$, and the winning payoff is $\sqrt{m}$. (A losing game earns nothing.) W seek the probability of winning more than $51.41 . The exact solution is obtained by enumerating outcomes.

PROBLEM D. X_D denotes the time from arrival until service begins for a customer in an M/G/1 queue characterized by an arrival rate of one per minute, and a service duration of 40 seconds exactly in 90% of the cases, and 2 minutes exactly for the remaining 10%. (Service times are assumed to be mutually independent.) The transform, mean and variance of X_D are available from the P-K formula of queueing theory. We seek the probability that a customer must wait longer than 11.61 minutes. No method (other than transform inversion) is available to compute the exact solution. In problems of this kind, simulation is often attempted, but simulation, by its very nature, cannot efficiently provide good estimates of small probabilities. The solution shown in Table 1 was obtained by the method given in this paper.

The values of U and L in Table 1 were chosen to span twice the effective range of each random variable. In problem D, the effective range was determined by quickly estimating tail probabilities with high precision but low accuracy to show that $P[X_D > 30] << 1$.

Figure 3 shows the relationship between accuracy error E_a and the approximate tail probability, τ, obtained when E_p and E_t are vanishingly small. Note that, for each problem, τ had come close to its limiting value by the time E_a reached $10^{-1.5} \sim 0.3$. The rate of convergence depends on the smoothness of the tail probability function - see Figure 1. For Problems A, C and D, τ approached the true tail probability (as $E_a \to 0$). In the case of Problem B, it approached $P[X>70] + 0.5\ P[X=70]$, which is not strictly correct, but satisfies (3.3) for any $E_a>0$. Note that the limiting values for Problems A and B nearly coincide, as predicted by the Central Limit Theorem, but that B converges more slowly, presumably because of the discontinuities in its tail probability function.

Figure 4 shows the truncation error $|\tau-\theta|$ as a function of the normalized computational effort z_t - see (3.9). The bound on truncation error (3.8) is also shown in the upper right corner of each graph. Case A converges the fastest - because the magnitude of its transform decreases rapidly as the index n in (1.3) increases. Case B remains constant over many iterations before changing, and then goes through a few intermediary stages with relatively large error. This demonstrates that the sum (1.3) cannot safely be terminated after its terms have remained small for a long time; they may suddenly increase periodically!

Note particularly that τ approaches θ, not the true tail probability, as N increases. Quick convergence in the sense of Figure 4 does not imply that a good approximation has been obtained.

7. CONCLUSIONS

The problem of determining a tail probability $P[X>A]$, given only the transform of X, has been shown to be NP-hard, implying that a computationally efficient algorithm to solve it exactly probably does not exist. Instead, we propose to use the simple approximation τ given by (1.3)-(1.4). The closeness of this approximation is given by (3.3). Eqs. (3.7)-(3.8) show how the "precision" errors can be made arbitrarily small, and (3.10) expresses the tradeoff between "accuracy" errors and computational effort.

We anticipate that this method will be particularly useful when computational resources are scarce and rough estimates of the tail probability will suffice, notably, when the threshold A is not exactly known, or when tail probabilities are required to evaluate alternatives in subroutines of large-scale optimization programs. However, when "exact" solutions are called for, our method will generally perform at least as well as, and certainly more efficiently than, others currently available.

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APPENDIX A

DERIVATION OF THE FOURIER SERIES (3.2)

Let

$$g(y) = \begin{cases} 1, \text{ if } y\text{-}L \bmod U\text{-}L > A\text{-}L \bmod U\text{-}L. \\ 0, \text{ if } y\text{-}L \bmod U\text{-}L < A\text{-}L \bmod U\text{-}L. \end{cases} \tag{A.1}$$

Since $g(\cdot)$ is periodic, it is given by the exponential Fourier series

$$g(y) = \sum_{n=-\infty}^{\infty} C_n \, e^{n\omega yj} \, , \tag{A.2}$$

where

$$C_n = \frac{1}{U-L} \int_L^U e^{-n\omega yj} \, g(y) \, dy = \frac{\beta^n - \gamma^n}{n2\pi j} \, , \qquad n = \pm 1, \pm 2, \ldots \, ,$$

$$C_0 = \frac{1}{U-L} \int_L^U g(y) \, dy = \frac{U-A}{U-L} \, , \tag{A.3}$$

and β, γ and ω are given by (1.4). If Y is any random variable, then by (A.2) and (1.1),

$$E[g(Y)] = \sum_{n=-\infty}^{\infty} C_n \, E\left[e^{n\omega jY} \right] = \sum_{n=-\infty}^{\infty} C_n \, \mathcal{L}_Y(-n\omega j). \tag{A.4}$$

Moreover, if Y is continuously distributed, then we may neglect the points of discontinuity of $g(\cdot)$, and (A.1) implies

$$E[g(Y)] = \text{Prob}[Y\text{-}L \bmod U\text{-}L > A\text{-}L \bmod U\text{-}L]. \tag{A.5}$$

Let $Y = X-DZ$, where X, D and Z are as in (3.1). Y is continuously distributed, so (A.5) holds. By (3.1) and (A.5),

$$\theta = E[g(Y)], \tag{A.6}$$

and by (A.4) and (A.6),

$$\theta = \sum_{n=-\infty}^{\infty} C_n \mathscr{L}_Y(-n\omega j). \tag{A.7}$$

Since $Y=X-DZ$, (1.2) implies $\mathscr{L}_Y(s) = \mathscr{L}_X(s) \cdot \mathscr{L}_{-DZ}(s)$, and so (A.7) becomes

$$\theta = \sum_{n=-\infty}^{\infty} C_n \mathscr{L}_X(-n\omega j) \mathscr{L}_{-DZ}(-n\omega j). \tag{A.8}$$

Moreover $-DZ$ is normally distributed with mean 0 and variance D^2, so $\mathscr{L}_{-DZ}(s) = e^{D^2 s^2/2} = \alpha^{n^2}$, where α is defined by (1.4). Substituting this into (A.8) yields

$$\theta = \sum_{n=-\infty}^{\infty} \alpha^{n^2} C_n \mathscr{L}_X(-n\omega j). \tag{A.9}$$

By (1.1), $\mathscr{L}(s^*) = \mathscr{L}^*(s)$ and by (A.3), $C_{-n}=C_n^*$ (where $*$ denotes complex conjugate). So (A.9) may be written as

$$\theta = C_0 + 2 \sum_{n=1}^{\infty} \alpha^{n^2} \operatorname{real}(C_n \mathscr{L}_X(-n\omega j)). \tag{A.10}$$

Substituting (A.3) into (A.10) produces (3.2).

APPENDIX B

PROOF OF THEOREM 1

By (3.1),

$$\theta \le P[X-DZ>A \text{ or } X-DZ<L] \le P[X>A+DZ] + P[X<L+DZ],$$
$$\theta \ge P[X-DZ\ge A \text{ and } X-DZ\le U] = P[X\ge A+DZ] - P[X>U+DZ], \qquad \text{(B.1)}$$

Now

$$P[X>A+DZ] = P[X>A+DZ|Z>-z_p]\ P[Z>-z_p] + P[X>A+DZ|Z<-z_p]\ P[Z<-z_p]$$
$$\le P[X>A+DZ|Z>-z_p] + P[Z<-z_p]$$
$$\le P[X>A-Dz_p] + \bar{\Phi}(z_p).$$

Proceeding similarly to bound the remaining probabilities on the right-hand side of (B.1), we obtain

$$\theta \le P[X>A-Dz_p] + P[X<L+Dz_p] + 2\ \bar{\Phi}(z_p),$$
$$\theta \ge P[X\ge A+Dz_p] - P[X>U-Dz_p] - 2\ \bar{\Phi}(z_p). \qquad \text{(B.2)}$$

Now (B.2) and (3.6)-(3.7) imply the first part of (3.3).

We now turn our attention to the second part of (3.3). By (1.1) and (1.4),

$$||\mathscr{L}_X(-n\omega j)|| \le 1, \qquad ||\beta|| \le 1, \qquad ||\gamma|| \le 1. \qquad \text{(B.3)}$$

By (3.2) and (1.3),

$$\tau - \theta = \sum_{n=N+1}^{\infty} \frac{\alpha^{n^2}}{n\pi} \cdot \text{Im}\Big[(\beta^n - \gamma^n) \cdot \mathscr{L}_X(-n\omega j) \Big] \qquad \text{(B.4)}$$

Now

$|\tau - \theta|$

$$\leq \sum_{n=N+1}^{\infty} \frac{\alpha^{n^2}}{n\pi} \cdot \left(\|\beta^n\| + \|\gamma^n\| \right) \cdot \|\mathcal{L}_X(-n\omega j)\| \qquad \text{(by (B.4))}$$

$$\leq \sum_{n=N+1}^{\infty} \frac{2}{n\pi}\, \alpha^{n^2} \qquad \text{(by (B.3))}$$

$$\leq \frac{2}{N\pi} \sum_{n=N+1}^{\infty} \alpha^{n^2} \qquad \text{(since } n>N \text{ over the range of summation)}$$

$$\leq \frac{2}{N\pi} \int_N^{\infty} \alpha^{n^2}\, dn \qquad \text{(by } \textit{monotonicity of the integrand}\text{)}$$

$$= \frac{2}{N\pi} \int_N^{\infty} e^{-(2\pi r n)^2/2}\, dn \qquad \text{(by (1.4) and (3.4))}$$

$$= \frac{2}{2\pi r N\pi} \int_{2\pi r N}^{\infty} e^{-z^2/2}\, dz \qquad (z = 2\pi r n)$$

$$= \frac{2\sqrt{2\pi}}{z_t \pi}\, \bar{\Phi}(z_t) \qquad \text{(by (3.5) and (3.9))}$$

$$< E_t\,. \qquad \text{(by (3.8))}$$

This establishes the second part of (3.3).

problem	range	mean	standard deviation	transform	L	U	A	exact solution
A	$-\infty$ to ∞	0	1	$e^{s^2/2}$	-10	10	4	0.3167×10^{-4}
B	0 to 100	50	5	$\left[\frac{e^{-s}+1}{2}\right]^{100}$	-50	150	70	0.1608×10^{-4}
C	0 to 61.67	20	7.853	$\prod_{m=1}^{20}\left[1+\frac{e^{-\sqrt{m}s}-1}{\sqrt{m}}\right]$	-30	92	51.41	0.7145×10^{-6}
D	0 to ∞	2	2.404	$\frac{0.2\,s}{s-1+0.9e^{-(2/3)s}+0.1e^{-2s}}$	-15	45	11.61	0.7037×10^{-2}

TABLE 1. *Parameters associated with numerical examples A though D.*

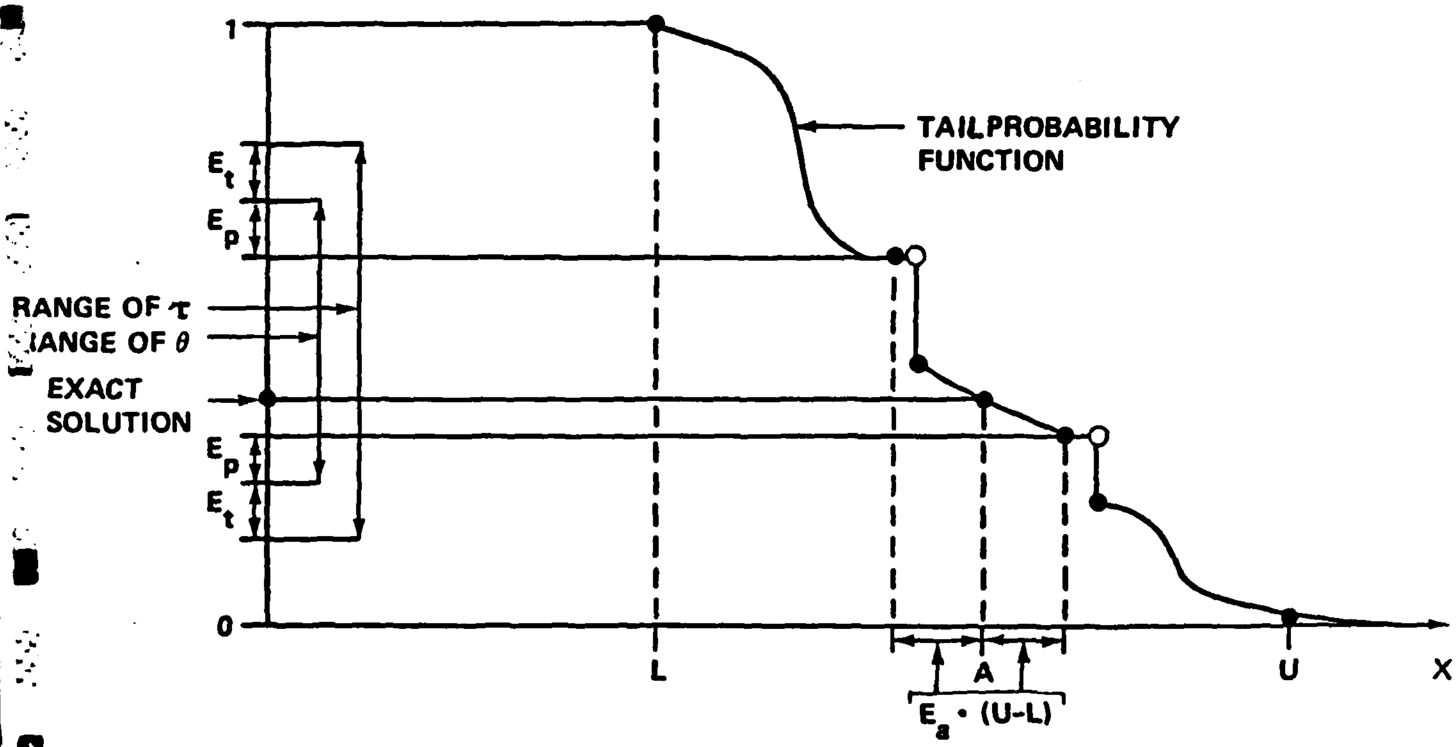

Figure 1. Geometric interpretation of E_a, E_p and E_t.

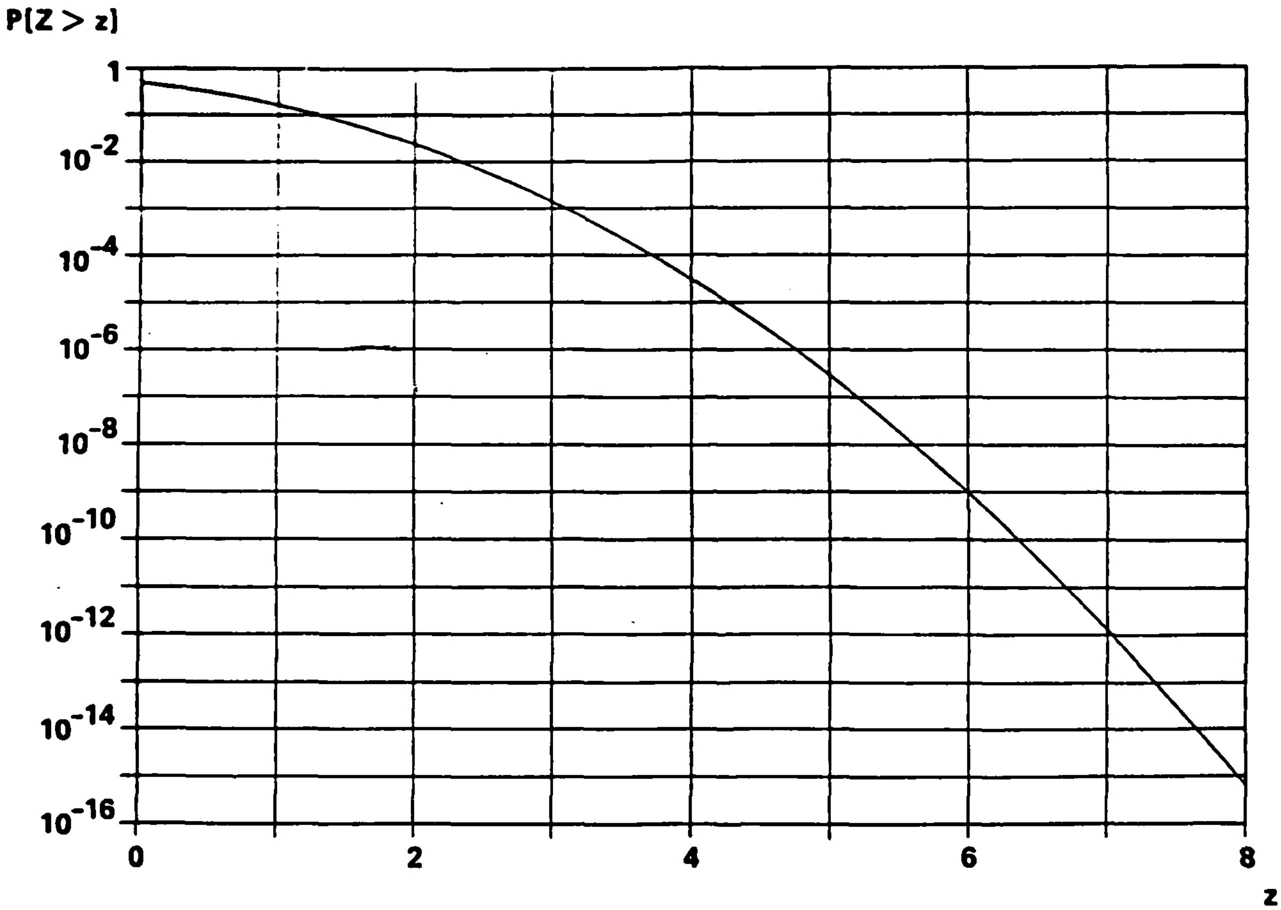

Figure 2. Tail probabilities of the normal distribution. (Calculated by the method given in this paper.)

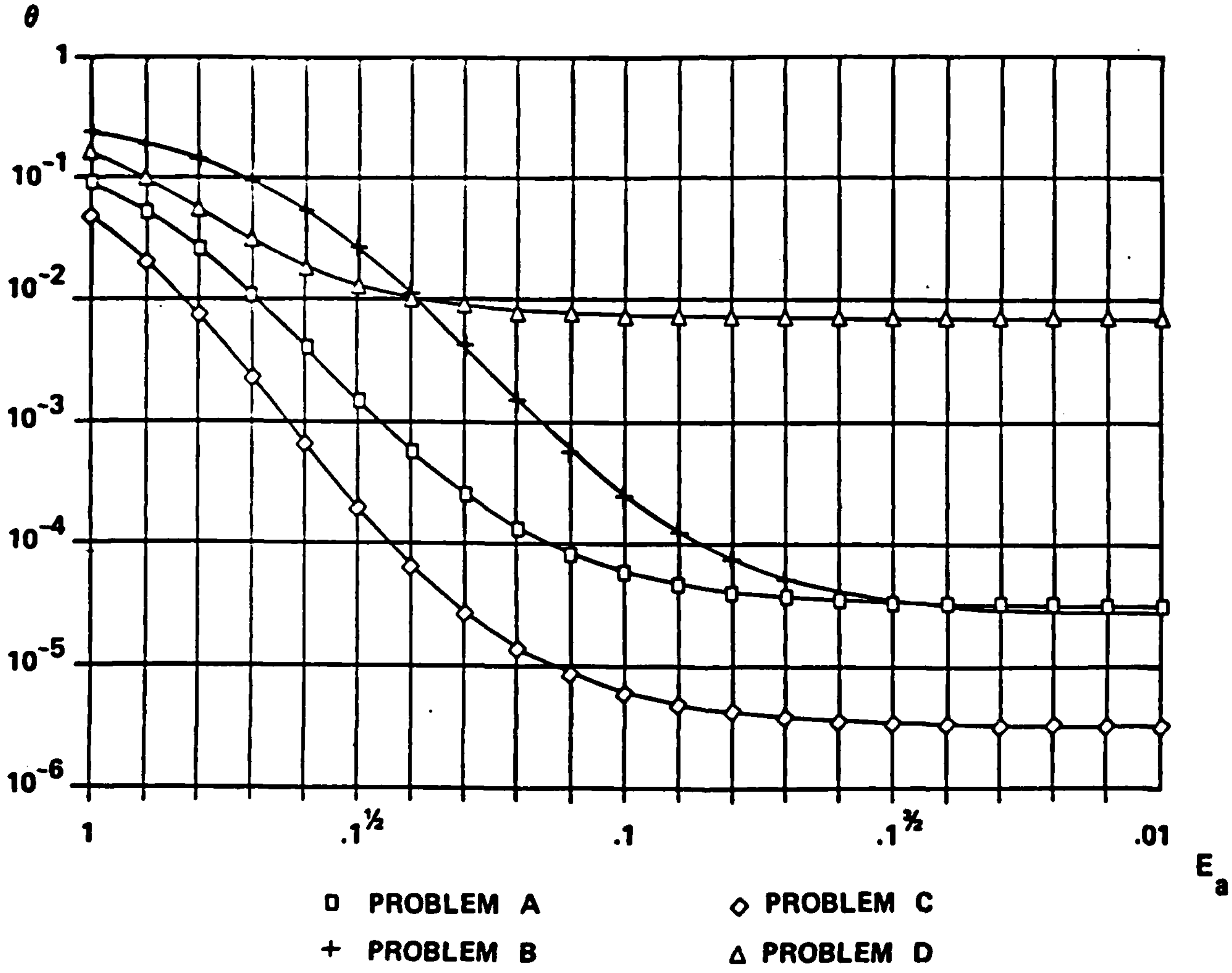

Figure 3. Approximate tail probability τ as a function of E_a when $z_p = z_t = 7$.

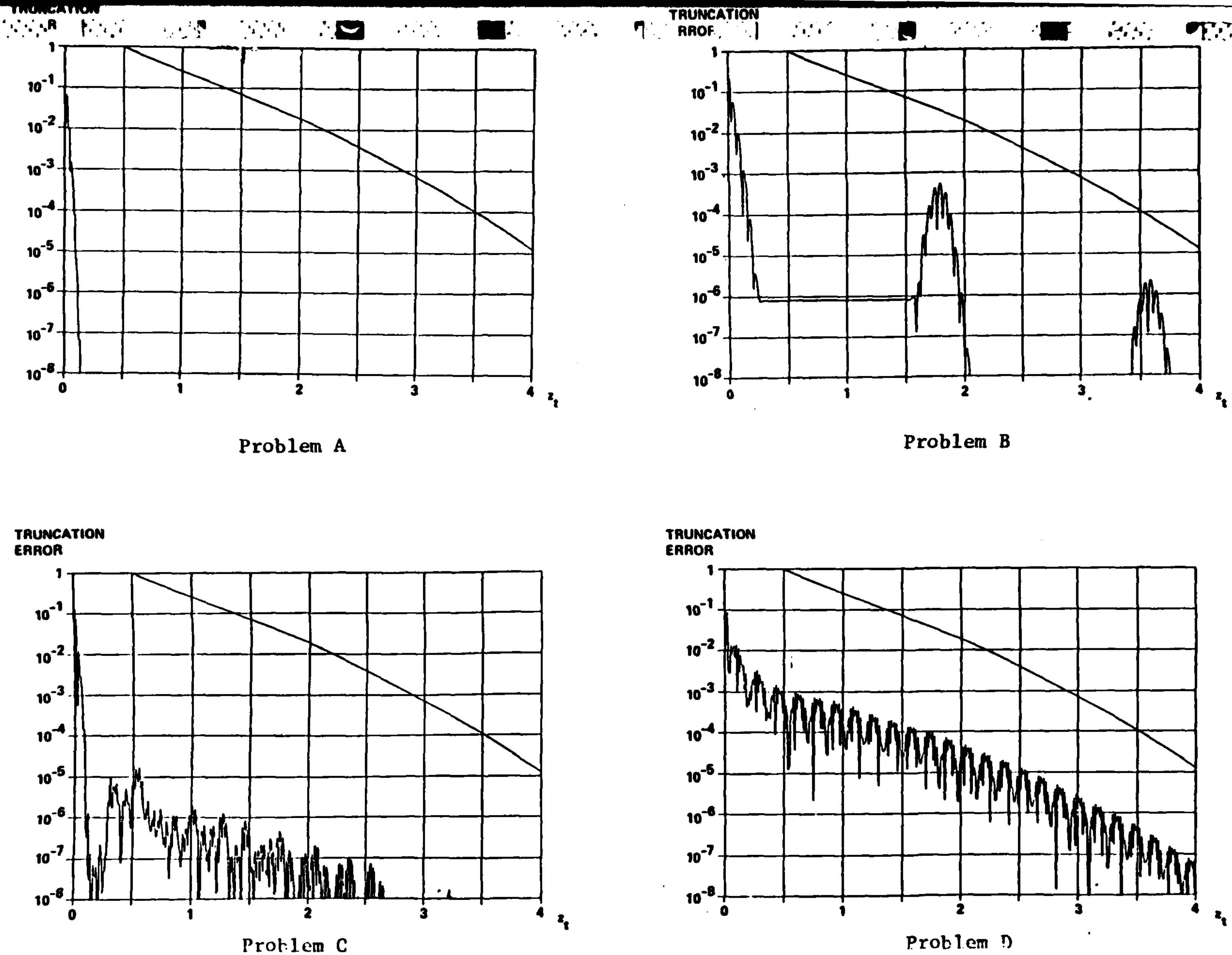

Figure 4. Truncation error $|\tau-\theta|$ as a function of normalized computational effort z_t when $z_p=7$ and $E_a=.01$.

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